Divergence operator and Poincaré inequalities on arbitrary bounded domains

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Abstract

Let Ω be an arbitrary bounded domain of $\mathbb{R}^n$. We study the right invertibility of the divergence on Ω in weighted Lebesgue and Sobolev spaces on Ω, and rely this invertibility to a geometric characterization of Ω and to weighted Poincaré inequalities on Ω. We recover, in particular, well-known results on the right invertibility of the divergence in Sobolev spaces when Ω is Lipschitz or, more generally, when Ω is a John domain, and focus on the case of s-John domains.

Contents

1 Introduction

Let $n \geq 2$. Throughout this paper, Ω will denote a nonempty bounded domain (*i.e.* an open connected subset of $\mathbb{R}^n$), and, for all $p \in [1, +\infty]$, $L_0^p(\Omega)$ stands for the subspace of $L^p(\Omega)$ made of functions having zero integral on Ω.
Let $1 \leq p \leq +\infty$. Does there exist a constant $C > 0$ such that, for all $f \in L_0^p(\Omega)$, one can find a vector field $u \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ such that div $u = f$ in Ω and

$$\|Du\|_{L^p(\Omega, \mathbb{R}^n)} \leq C \, \|f\|_{L^p(\Omega)}?$$

This problem arises in various contexts. In particular, it plays a key role (for $p = 2$) in the analysis of the Stokes problem by finite elements methods ([8], Chapter 10). It is also involved in the study of Hardy-Sobolev spaces on strongly Lipschitz domains of $\mathbb{R}^n$ ([3]).
When $p = 1$ or $p = +\infty$, the problem has no solution in general, even in $W^{1,p}(\Omega)$ (see [9] when $p = 1$ and [18] when $p = +\infty$).
When $1 < p < +\infty$ and Ω is Lipschitz, the question receives a positive answer. A reformulation of this result is that the divergence operator, mapping $W_0^{1,p}(\Omega, \mathbb{R}^n)$ to $L_0^p(\Omega)$, has a continuous right inverse.
Several approaches can be found for this theorem in the literature. It can for instance be proved by functional analysis arguments (see [3, 19]), or by a somewhat elementary construction ([7]). In dimension 2 and when Ω is smooth (or is a convex polygon), it can be solved via the Neumann problem for the Laplace operator on Ω ([4, 8, 14, 17]). This approach can also be adapted to the case when Ω is a non-convex polygon, relying on suitable trace theorems ([2]).

A different and more explicit construction, valid in star-shaped domains with respect to a ball and for weighted norms, was developed in [6] and in [13]. The idea of this construction is to integrate test functions on segments included in Ω, which yields a somewhat explicit formula for u. This approach was extended to John domains in [1], replacing segments by appropriate rectifiable curves included in Ω.

However, there exist domains Ω such that div $: W_0^{1,p}(\Omega) \to L_0^p(\Omega)$ has no continuous right inverse. A counterexample, originally due to Friedrichs, can be found in [1] and is, actually, a planar domain with an external cusp of power type. In the present paper, we investigate the solvability of div $u = f$ on *arbitrary* bounded domains, with estimates in weighted Lebesgue or Sobolev spaces.
More precisely, we consider the following question: does there exist an integrable weight $\mathrm{w} > 0$ on Ω such that the divergence operator, acting from $L^p\left(\Omega, \frac{1}{\mathrm{w}} dx\right)$ to $L^p(\Omega, dx)$, has a continuous right inverse ? When $p = +\infty$, we give a necessary and sufficient condition on Ω for this property to hold. This condition says that the geodesic distance in Ω to a fixed point is

integrable over Ω. Under this assumption, the previous divergence problem is also solvable in L^p norms for all $p \in (1, +\infty)$.

The proofs rely on two key tools: first, we extend the method developed in [6, 13, 1] to the case of an arbitrary bounded domain, using adapted rectifiable curves inside the domain. Then, we show that the solvability of the divergence problem in L^p is equivalent to a weighted Poincaré inequality in L^q with $\frac{1}{p} + \frac{1}{q} = 1$.

When the divergence problem is solvable in L^p spaces, we also solve it in weighted Sobolev spaces on Ω. This extends to the case of arbitrary bounded domains the previously cited theorems when Ω is Lipschitz or, more generally, is a John domain. A key tool for this, interesting in itself, is a kind of atomic decomposition for functions in $L^p(\Omega)$ with zero integral. We also obtain interesting and new results for the invertibility of the divergence on Sobolev spaces in the class of s-John domains and, among these domains, in the sub-class of strongly hölderian domains. It should be noted that solvability results in weighted Sobolev spaces for some hölderian planar domains had already been obtained by the first author and F. Lopez Garcia in [12].

In the case when $p = 1$, as we said, it is impossible to solve div $u = f$ when $f \in L^1(\Omega)$ with $u \in W^{1,1}(\Omega)$ in general. However, if f is supposed to be in a Hardy space on Ω instead of belonging to $L^1(\Omega)$, then, under suitable assumptions on Ω, it is possible to solve in appropriate Hardy-Sobolev spaces, considered in [3]. We will come back to this issue in a forthcoming paper.

Here is the outline of the paper. In Section 2, we establish the equivalence between the solvability of the divergence in L^∞ spaces and the integrabilty of the geodesic distance to a fixed point. Section 3 is devoted to the solvability in L^p spaces and the equivalence with L^q Poincaré inequalities. In Section 4, we investigate the solvability of the divergence in weighted Sobolev spaces. Finally, in Section 5, we focus on the case of s-John domains and, in particular, on the class of strongly hölderian domains.

2   Invertibility of the divergence in L^∞ spaces

Throughout this section, Ω denotes a bounded domain (connected open subset) of $\mathbb{R}^n$, without any assumption on the regularity of its boundary $\partial\Omega$. We investigate how to invert the divergence operator on L^∞ spaces, asking the following question: does it exist a weight w, defined on Ω, such that

   $i)$ w is integrable over Ω (*i.e.* w $\in L^1(\Omega)$),

ii) for each $f \in L^\infty(\Omega)$ with $\int_\Omega f(x)dx = 0$, there exists a vector-valued function u solution of

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega, \\ u \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

satisfying the estimate

$$\left\| \frac{1}{\mathrm{w}} u \right\|_\infty \leq C \|f\|_\infty, \tag{2.2}$$

where $C > 0$ only depends on Ω and the weight w?

We define u being a solution of (2.1) by demanding that

$$\int_\Omega u \cdot \nabla g = -\int_\Omega fg \tag{2.3}$$

for all g in the space E of those functions in $L^1(\Omega)$ such that their distributional gradient is a function and

$$\|u\|_E = \int_\Omega |g| + \int_\Omega \mathrm{w} \, |\nabla g| < +\infty.$$

The integrability condition *i*) above ensures that $\mathcal{D}(\overline\Omega)$, the set of restrictions to Ω of test functions in $\mathbb{R}^n$, is embedded in E. Thus, equation (2.3) defines the meaning of both the partial differential equation and the boundary condition in (2.1).

We say that the problem $(\operatorname{div})_\infty$ is solvable in Ω when this question receives a positive answer.

We introduce $\operatorname{dist}_\Omega$, the euclidean geodesic distance in Ω, defined by

$$\operatorname{dist}_\Omega(y, x) = \inf l(\gamma),$$

where γ is any rectifiable path defined on $[0, 1]$ such that $\gamma(0) = y$, $\gamma(1) = x$, and $\gamma(t) \in \Omega$ for all $t \in [0, 1]$, $l(\gamma)$ being its length. We pick once and for all a reference point $x_0 \in \Omega$, and we simply denote by $d_\Omega(y)$ the number $\operatorname{dist}_\Omega(y, x_0)$.

Our main result is the following:

Theorem 2.1 *The problem $(div)_\infty$ is solvable in Ω if and only if $d_\Omega \in L^1(\Omega)$.*

Notice that d_Ω being integrable on Ω does not depend on the choice of the reference point x_0.

To prove this theorem, assume first that $(\operatorname{div})_\infty$ is solvable in Ω. We will explain in the next section that it is equivalent to various Poincaré inequalities, implying in particular the following one:

$$\int_\Omega |f(x)| \, dx \leq C \int_\Omega \mathrm{w}(x) \, |\nabla f(x)| \, dx$$

for all $f \in \mathrm{W}^{1,1}_{loc}(\Omega)$ which vanishes on a non negligible subset K of Ω, the constant C depending on Ω and on the ratio $\frac{|\Omega|}{|K|}$.

We apply this inequality to

$$f(x) = (d_\Omega(x) - r_0)_+,$$

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where r_0 is the radius of some fixed ball centered at x_0 and included in Ω. Since

$$|d_\Omega(y) - d_\Omega(x)| \leq |y - x|$$

for all close enough $x, y \in \Omega$, we have $|\nabla f(x)| \leq 1$ almost everywhere, so that

$$\int_\Omega \mathrm{w}(x) \, |\nabla f(x)| \, dx \leq \int_\Omega \mathrm{w}(x) dx < +\infty.$$

Thus $f \in L^1(\Omega)$, which implies that d_Ω is integrable on Ω, too.

Reciprocally we assume that

$$\int_\Omega d_\Omega(y) dy < +\infty, \tag{2.4}$$

and explain how to solve $(\mathrm{div})_\infty$ under this hypothesis. Our method will be constructive.

The starting point is a family of curves relying each point in Ω to x_0. For the sake of simplicity, the image of any path γ will be called γ, too.

We use a Whitney decomposition of Ω, whose properties we recall. It is, up to negligible sets, a partition of Ω in closed cubes:

$$\Omega = \bigcup_{j \geq 0} Q_j,$$

where the interiors of the Q_j's are pairwise disjoint, and for all $j \geq 0$, $2Q_j \subset \Omega$ while $4Q_j \cap \Omega^c \neq \emptyset$. We denote by x_j and l_j the center and the side length of Q_j (we may and do assume that x_0 is the center of Q_0). A key observation of constant use is that, if $x \in \Omega$, then

$$\frac{1}{2} l_j \leq d(x) \leq \frac{5\sqrt{n}}{2} l_j$$

whenever $x \in Q_j$. We denote by $d(x)$ the distance from x to the boundary of Ω.

We select a path $\gamma_j : [0,1] \mapsto \Omega$ such that $\gamma_j(0) = x_j$, $\gamma_j(1) = x_0$, $l(\gamma_j) \leq 2d_\Omega(x_j)$. When γ_j intersects some Whitney cube Q_k, we replace $\gamma_j \cap Q_k$ by the segment joining its two endpoints. In particular, we call $[x_j, \widetilde{x}_j]$ the segment $\gamma_j \cap Q_j$. The modified path is still denoted by γ_j. We certainly have not increased its length, and moreover we have ensured that

$$l(\gamma_j \cap B(x, r)) \leq Cr$$

for every $x \in \Omega$ and $r \leq \frac{1}{2} d(x)$, where C only depends on the dimension.

Finally, if $y \in Q_j$ for some j (when there are several such j's, just arbitrarily select one of them), we link y to $\widetilde{x}_j$ by a segment, and then $\widetilde{x}_j$ to x_0 by γ_j. The resulting path linking y to x_0 is called $\gamma(y)$, and its current point $\gamma(t, y)$, where $t \in [0,1]$. By construction, the following properties hold:

$(\gamma.a)$ for all $y \in \Omega$ and $t \in [0,1]$, $\gamma(t, y) \in \Omega$, with $\gamma(0, y) = y$, $\gamma(1, y) = x_0$,

$(\gamma.b)$ $(t, y) \mapsto \gamma(t, y)$ is measurable and $\gamma(y)$ is rectifiable,

$(\gamma.c)$ there exists a constant $C > 0$, only depending on the dimension, such that for all $x, y \in \Omega$

$$\forall r \le \frac{1}{2}d(x), \ l(\gamma(y) \cap B(x, r)) \le Cr \tag{2.5}$$

and

$$l(\gamma(y)) \le Cd_\Omega(y), \tag{2.6}$$

$(\gamma.d)$ for all $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that

$$\forall y \in \Omega_\varepsilon, \ \gamma(y) \subset \Omega_\delta,$$

where by definition $\Omega_s = \{z \in \Omega; d(z) > s\}$.

We now define a weight ω on Ω (implicitly depending on the family of paths γ) by

$$\omega(x) = \left| \left\{ y \in \Omega; \ \text{dist}_\Omega(\gamma(y), x) \le \frac{1}{2}d(x) \right\} \right|, \tag{2.7}$$

where $\text{dist}_\Omega(\gamma(y), x)$ is the distance in Ω from the path $\gamma(y)$ to the point x. We first prove that:

Lemma 2.2

$$\int_\Omega \omega(x)d(x)^{-n+1}dx < +\infty. \tag{2.8}$$

Proof: Let y be such that there exists t_0 satisfying $\text{dist}_\Omega(\gamma(t_0, y), x) \le \frac{1}{2}d(x)$, which here is the same as $|\gamma(t_0, y) - x| \le \frac{1}{2}d(x)$. Then we have $|\gamma(t, y) - x| < \frac{3}{4}d(x)$ whenever $|\gamma(t, y) - \gamma(t_0, y)| < \frac{1}{4}d(x)$. This implies that

$$\begin{aligned} d(x) &\le 4l(\gamma(y) \cap B(\gamma(t_0, y), \tfrac{1}{4}d(x))) \\ &\le 4 \int_0^1 |\dot\gamma(t, y)| \, \mathbf{1}_{|\gamma(t,y)-x|<\frac{3}{4}d(x)} dt, \end{aligned}$$

and therefore that

$$\int_\Omega \omega(x)d(x)^{-n+1}dx \le C \int_\Omega \int_\Omega \int_0^1 |\dot\gamma(t, y)| \, \mathbf{1}_{|x-\gamma(t,y)|<\frac{3}{4}d(x)} d(x)^{-n} dt dy dx.$$

In the integral above, $d(x)$ is comparable with $d(\gamma(t, y))$ uniformly in x, t and y. Integrating first with respect to x, this observation leads to

$$\begin{aligned} \int_\Omega \omega(x)d(x)^{-n+1}dx &\le C \int_\Omega \int_0^1 |\dot\gamma(t, y)| \, dt dy \\ &= C \int_\Omega l(\gamma(y))dy, \end{aligned}$$

and the proof is ended thanks to the hypothesis (2.4) and the estimate (2.6).

The key result is then the following, that we state in full generality for further use:

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Lemma 2.3 *Assume that Ω fulfills(2.4). Let $\gamma = \{\gamma(y), y \in \Omega\}$ be any family of paths satisfying properties ($\gamma.a$) to ($\gamma.d$), and ω defined by (2.7). For each $f \in L^\infty(\Omega)$ with $\int_\Omega f(x)dx = 0$, there exists a vector-valued function u solution of*

$$\begin{cases} div\, u = f & in\ \Omega, \\ u \cdot \nu = 0 & on\ \partial\Omega, \end{cases} \tag{2.9}$$

satisfying the estimate

$$\left\| \frac{d^{n-1}}{\omega} u \right\|_\infty \leq C \|f\|_\infty, \tag{2.10}$$

where $C > 0$ only depends on Ω and the choice of the family γ.

Proof: it relies on a representation formula, which goes back (in its earlier version) to [6]. A first generalization, applicable to John domains, was designed in [1]. Here, we still generalize it to the case of an arbitrary domain.

By dilating and translating Ω, we may and do assume that $x_0 = 0$ and $d(0) = 15$. We choose a function $\chi \in \mathcal{D}(\Omega)$ supported in $\overline{B(0,1)}$ and such that $\int_\Omega \chi(x)dx = 1$. For each $y \in \Omega$, let $\tau(y)$ be the smallest $t > 0$ such that $\gamma(t, y) \in \partial B(y, \frac{1}{2}d(y))$ (if there is no such t, just choose $\tau(y) = 1$). We define a function $t \mapsto \rho(t, y)$, $t \in [0, 1]$, by:

$$\begin{aligned} \rho(t, y) &= \alpha\, |y - \gamma(t, y)| && \text{if } t \leq \tau(y), \\ \rho(t, y) &= \tfrac{1}{15}d(\gamma(t, y)) && \text{if } t > \tau(y), \end{aligned}$$

where α is so chosen that $\rho(\cdot, y)$ is a continuous function. This means that

$$\alpha = \frac{2}{15}\frac{d(\gamma(\tau(y), y))}{d(y)},$$

and the reader may check that $\alpha \leq \frac{1}{5}$. By construction, we always have

$$\rho(t, y) \leq \frac{1}{5}d(\gamma(t, y)) \tag{2.11}$$

so that

$$\gamma(t, y) + \rho(t, y)z \in \Omega$$

for every $t \in [0, 1]$ and $z \in B(0, 1)$. This comes from the fact that, if $t \leq \tau(y)$, then $|y - \gamma(t, y)| \leq \frac{1}{2}d(y)$, which implies $\rho(t, y) \leq \frac{\alpha}{2}d(y)$ and $d(y) \leq 2d(\gamma(t, y))$.

Let us start with $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $y \in \Omega$ and $z \in B(0, 1)$. We have

$$\varphi(y) - \varphi(z) = -\int_0^1 (\dot{\gamma}(t, y) + \dot{\rho}(t, y)z) \cdot \nabla\varphi(\gamma(t, y) + \rho(t, y)z)dt.$$

Multiplying by $\chi(z)$ and integrating, we get

$$\varphi(y) - \int_\Omega \varphi\chi = -\int_{B(0,1)}\int_0^1 (\dot{\gamma}(t, y) + \dot{\rho}(t, y)z) \cdot \nabla\varphi(\gamma(t, y) + \rho(t, y)z)\chi(z)dtdz.$$

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Since $\int_\Omega f = 0$, this implies

$$\int_\Omega f\varphi = -\int_\Omega \int_{B(0,1)} \int_0^1 f(y)(\dot\gamma(t,y) + \dot\rho(t,y)z) \cdot \nabla\varphi(\gamma(t,y) + \rho(t,y)z)\chi(z)dtdzdy.$$

Changing z into $x = \gamma(t,y) + \rho(t,y)z$, this formula becomes

$$\int_\Omega f\varphi = -\int_\Omega \int_\Omega \int_0^1 f(y)\left(\dot\gamma(t,y) + \dot\rho(t,y)\frac{x - \gamma(t,y)}{\rho(t,y)}\right)\cdot\nabla\varphi(x)\chi\left(\frac{x - \gamma(t,y)}{\rho(t,y)}\right)\frac{1}{\rho(t,y)^n}dtdxdy.$$
(2.12)

If $x,y \in \Omega$, we define the vector-valued kernel G by

$$G(x,y) = \int_0^1 \left[\dot\gamma(t,y) + \dot\rho(t,y)\frac{x - \gamma(t,y)}{\rho(t,y)}\right]\chi\left(\frac{x - \gamma(t,y)}{\rho(t,y)}\right)\frac{dt}{\rho(t,y)^n}.$$

Thanks to the support condition on χ, we must have

$$|x - \gamma(t,y)| < \rho(t,y)$$

for the integrated term to be non zero. Hence $G(x,y)$ is well defined as soon as $x \neq y$. We moreover have

Lemma 2.4 *There exists a constant $C > 0$, depending on Ω and on the paths γ only, such that*

$$\forall x \in \Omega, \quad \int_\Omega |G(x,y)|\,dy \leq C\omega(x)d(x)^{-n+1}.$$
(2.13)

Let us admit this statement for the moment and finish the proof of Lemma 2.3. We define u by

$$u(x) = \int_\Omega G(x,y)f(y)dy.$$

The estimate (2.13) shows that this is meaningful and that

$$|u(x)| \leq C \|f\|_\infty \omega(x)d(x)^{-n+1}.$$
(2.14)

Thus (2.10) is satisfied. Then, it follows from (2.12) and Fubini theorem, which we may apply thanks to Lemma 2.2 and (2.13), that

$$\int_\Omega u \cdot \nabla\varphi = -\int_\Omega f\varphi;$$
(2.15)

this is (2.3) for $\varphi \in \mathcal{D}(\overline\Omega)$.
Let us now take $g \in E$. If $\varepsilon > 0$ is small enough, we define

$$f_\varepsilon = f\mathbf{1}_{\Omega_\varepsilon}$$

and

$$u_\varepsilon(x) = \int_\Omega G(x,y)f_\varepsilon(y)dy.$$

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We have $u_\varepsilon(x) \to u(x)$ for all $x \in \Omega$, and

$$|u_\varepsilon(x)| \leq C \|f\|_\infty \omega(x)d(x)^{-n+1}, \tag{2.16}$$

by Lemma 2.4. Because of property $(\gamma.d)$ there exists $\delta > 0$ such that Supp $u_\varepsilon \subset \Omega_\delta$. Moreover, there exists $C(\delta)$ satisfying

$$\forall x \in \Omega_\delta \ \frac{1}{C(\delta)} \leq \omega(x)d(x)^{-n+1} \leq C(\delta).$$

Hence, we may use a standard approximation argument to deduce from (2.15), applied to f_ε and u_ε, that

$$\int_\Omega u_\varepsilon \cdot \nabla g = -\int_\Omega f_\varepsilon g.$$

We conclude by using (2.16) and the dominated convergence theorem to obtain:

$$\int_\Omega u \cdot \nabla g = -\int_\Omega fg$$

for all $g \in E$.

We turn to the proof of Lemma 2.4. Since ρ is defined according to two different cases, we decompose

$$G(x,y) = G_1(x,y) + G_2(x,y)$$

with

$$G_1(x,y) = \int_0^{\tau(y)} \left[\dot\gamma(t,y) + \alpha \frac{\dot\gamma(t,y) \cdot (y - \gamma(t,y))}{|y - \gamma(t,y)|} \right] \chi \left(\frac{x - \gamma(t,y)}{\alpha |y - \gamma(t,y)|} \right) \frac{1}{\alpha^n |y - \gamma(t,y)|^n} dt,$$

and

$$G_2(x,y) = \int_{\tau(y)}^1 \left[\dot\gamma(t,y) + [\dot\gamma(t,y) \cdot \nabla d(\gamma(t,y))] \frac{x - \gamma(t,y)}{d(\gamma(t,y))} \right] \chi \left(15 \frac{x - \gamma(t,y)}{d(\gamma(t,y))} \right) \frac{15^n}{d(\gamma(t,y))^n} dt. \tag{2.17}$$

We first estimate $\int_\Omega |G_1(x,y)| \, dy$. When $\chi(\frac{x-\gamma(t,y)}{\alpha|y-\gamma(t,y)|})$ is non zero, we have

$$\begin{aligned} |x - y| \ &\leq (1+\alpha) |y - \gamma(t,y)| \\ &\leq \tfrac{1+\alpha}{2} d(y) \\ &\leq \tfrac{1+\alpha}{1-\alpha} d(x) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} |x - \gamma(t,y)| \ &\leq \alpha |y - \gamma(t,y)| \\ &\leq \tfrac{\alpha}{1-\alpha} |x - y|. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} |G_1(x,y)| \ &\leq C \int_0^{\tau(y)} |\dot\gamma(t,y)| \, \mathbf{1}_{|x-\gamma(t,y)| \leq \frac{\alpha}{1-\alpha}|x-y|} dt \, |x - y|^{-n} \\ &\leq C |x - y|^{-n} l(\gamma(y) \cap B(x, \tfrac{\alpha}{1-\alpha} |x - y|)). \end{aligned}$$

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Since $\alpha \leq \frac{1}{5}$, we deduce from (2.18) that $\frac{\alpha}{1-\alpha} |x-y| \leq \frac{3}{8}d(x)$. We may therefore apply property $(\gamma.c)$, which gives

$$|G_1(x,y)| \leq C |x-y|^{-n+1} .$$

Applying (2.18) again, we see that

$$\int_\Omega |G_1(x,y)| \, dy \leq C d(x).$$

By definition of ω there is a constant c depending on the dimension such that $\omega(x) \geq c d(x)^n$, since $B(x, \frac{1}{2}d(x)) \subset \{y; \text{dist}_\Omega(\gamma(y), x) \leq \frac{1}{2}d(x)\}$. Thus the analog of inequality (2.13) is satisfied by G_1.

We now estimate $\int_\Omega |G_2(x,y)| \, dy$. We have from (2.17):

$$|G_2(x,y)| \leq C \int_{\tau(y)}^1 |\dot\gamma(t,y)| \, \mathbf{1}_{|x-\gamma(t,y)|<\frac{1}{15}d(\gamma(t,y))} \frac{dt}{d(\gamma(t,y))^n}.$$

On the domain of integration, the ratio $\frac{d(x)}{d(\gamma(t,y))}$ is bounded from above and below by $16/15$ and $14/15$. This implies the estimate

$$|G_2(x,y)| \leq C d(x)^{-n} l(\gamma(y) \cap B(x, \frac{1}{14}d(x))),$$

so that, by property $(\gamma.c)$, we have

$$|G_2(x,y)| \leq C d(x)^{-n+1}$$

uniformly in x, y, and $G_2(x,\cdot)$ is supported in the set $\{y; d(\gamma(y), x) < \frac{1}{14}d(x)\}$. We therefore obtain the analog of inequality (2.13) for G_2 as well, concluding the proof of Lemma 2.4.

3   Invertibility of the divergence in L^p spaces

The present section is devoted to the proof of the following L^p-type result:

Theorem 3.1 *Let $p \in (1, +\infty]$ and $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain. Assume that the function d_Ω is integrable on Ω. Then, if $f \in L^p(\Omega)$ with $\int_\Omega f(x)dx = 0$, there exists a vector-valued function u solution of*

$$\begin{cases} div\ u = f & in\ \mathcal{D}'(\Omega), \\ u \cdot \nu = 0 & on\ \partial\Omega, \end{cases} \tag{3.1}$$

satisfying the estimate

$$\left\| \frac{d^{n-1}}{\omega} u \right\|_p \leq C \|f\|_p, \tag{3.2}$$

where $C > 0$ only depends on Ω and the choice of the paths γ.

Let us clarify the meaning of (3.1). Define $q \in [1, +\infty)$ by $\frac{1}{p} + \frac{1}{q} = 1$ and let

$$E^q(\Omega) := \left\{ g \in L^q(\Omega) \; ; \; \int_\Omega |\nabla g(x)|^q \left(\frac{\omega(x)}{d^{n-1}(x)} \right)^q < +\infty \right\},$$

equipped with the norm

$$\|g\|_{E^q(\Omega)} := \|g\|_{L^q(\Omega)} + \left\| \frac{\omega(x)}{d^{n-1}(x)} |\nabla g| \right\|_{L^q(\Omega)}.$$

By (3.1), we mean that

$$\int_\Omega u \cdot \nabla \varphi = \int_\Omega f \varphi$$

for all $\varphi \in E^q(\Omega)$.

The proof of Theorem 3.1 goes through a family of Poincaré inequalities, which we present now. If $1 \le q < +\infty$, say that Ω supports a weighted L^q Poincaré inequality if and only if, for all function $g \in E^q(\Omega)$,

$$\|g - g_\Omega\|_{L^q(\Omega)} \le C \left\| \frac{\omega}{d^{n-1}} |\nabla g| \right\|_{L^q(\Omega)}, \qquad (P_q)$$

where

$$g_\Omega := \frac{1}{|\Omega|} \int_\Omega g(x) dx.$$

The strategy of the proof of Theorem 3.1 is as follows: we first establish the general fact that, for all $p \in (1, +\infty]$, the solvability of (3.1) for all $f \in L^p(\Omega)$ having zero integral with the estimate (3.2) is equivalent to the validity of (P_q), with $\frac{1}{p} + \frac{1}{q} = 1$. We then prove that (P_1) implies (P_q) for any $q \in [1, +\infty)$. It is then a consequence of Lemma 2.3 that, since $d_\Omega \in L^1(\Omega)$, (P_1), and therefore (P_q), hold for all $q \in [1, +\infty)$ and Theorem 3.1 therefore follows again from the equivalence with (P_q).

In the sequel, we will say that (div_p) holds if and only if, for all $f \in L^p(\Omega)$ with zero integral, there exists a vector-valued function u solution of (3.1) such that the estimate (3.2) holds with a constant $C > 0$ only depending on Ω and the choice of the paths γ.

3.1 Solvability of the divergence and Poincaré inequalities

Let us prove the following equivalence:

Proposition 3.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $1 < p \le +\infty$ and $1 \le q < +\infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then (P_q) holds if and only if (div_p) holds.*

Proof: Assume first that (div_p) holds. Let $f \in E^q(\Omega)$ and $g \in L^p(\Omega)$ with $\|g\|_p \le 1$. Pick up a solution u of

$$\begin{cases} \text{div } u = g - g_\Omega & \text{in } \Omega, \\ u \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

satisfying
$$\left\| \frac{d^{n-1}}{\omega} u \right\|_p \leq C \left\| g \right\|_p.$$

Then, by definition of u,

$$
\begin{aligned}
\left| \int_\Omega (f(x) - f_\Omega)\, g(x) dx \right| &= \left| \int_\Omega (f(x) - f_\Omega)\, (g(x) - g_\Omega)\, dx \right| \\
&= \left| \int_\Omega \nabla f(x) \cdot u(x) dx \right| \\
&\leq \left\| \frac{\omega}{d^{n-1}} |\nabla f| \right\|_q \left\| \frac{d^{n-1}}{\omega} u \right\|_p \\
&\leq C \left\| \frac{\omega}{d^{n-1}} |\nabla f| \right\|_q \left\| g \right\|_p \\
&\leq C \left\| \frac{\omega}{d^{n-1}} |\nabla f| \right\|_q,
\end{aligned}
$$

which proves that

$$\left\| f - f_\Omega \right\|_q \leq C \left\| \frac{\omega}{d^{n-1}} |\nabla f| \right\|_q,$$

and this means that (P_q) holds.

Assume conversely that (P_q) holds and let $f \in L^p(\Omega)$ with zero integral. Define

$$\mathcal{G}_q := \left\{ v \in L^q \left(\Omega, \mathbb{R}^n, \frac{\omega}{d^{n-1}} \right) ; \ v = \nabla g \text{ for some } g \in E^q(\Omega) \right\},$$

equipped with the norm of $L^q \left(\Omega, \mathbb{R}^n, \frac{\omega}{d^{n-1}} \right)$.
Define a linear form on $\mathcal{G}_q$ by

$$Lv = \int_\Omega f(x) g(x) dx \text{ if } v = \nabla g.$$

Observe that L is well-defined since f has zero integral on Ω and that

$$
\begin{aligned}
|L(v)| &= \left| \int_\Omega f(x)\, (g(x) - g_\Omega)\, dx \right| \\
&\leq C \left\| f \right\|_p \left\| \frac{\omega}{d^{n-1}} |\nabla g| \right\|_q
\end{aligned}
$$

since (P_q) holds; this shows that L is bounded on $\mathcal{G}_q$. By the Hahn-Banach theorem, L may be extended to a bounded linear form on $L^q \left(\Omega, \mathbb{R}^n, \frac{\omega}{d^{n-1}} \right)$ with functional norm bounded by $C \left\| f \right\|_p$. Therefore, since $1 \leq q < +\infty$, there exists a vector field $u \in L^p \left(\Omega, \mathbb{R}^n, \frac{\omega}{d^{n-1}} \right)$ such that, for all $g \in E^q(\Omega)$,

$$\int_\Omega u(x) \cdot \nabla g(x) dx = \int_\Omega f(x) g(x) dx, \tag{3.3}$$

with

$$\left\| \frac{d^{n-1}}{\omega} u \right\|_p \leq C \left\| f \right\|_p. \tag{3.4}$$

Note that, for $p = 1$, this uses the fact that the measure $\frac{\omega}{d^{n-1}} dx$ is finite (Lemma 2.2). The identity (3.3) means that u solves (3.1), and (3.4) is exactly (3.2). This ends the proof of Proposition 3.2. $\qquad\square$

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3.2 From an L^1 to an L^q Poincaré inequality

We now prove the following fact:

Proposition 3.3 *Assume that (P_1) holds. Then, for all $q \in [1, +\infty[$, (P_q) holds.*

The proof mimics a usual procedure for Poincaré inequalities (see [16]) and we give it for the sake of completeness. It relies on a very useful characterization of Poincaré inequalities:

Lemma 3.4 *Let $1 \leq q < +\infty$. Then, (P_q) holds if and only if there exists $C > 0$ with the following property: for all measurable subset $E \subset \Omega$ with $|E| \geq \frac{1}{2} |\Omega|$ and for all $g \in E^q(\Omega)$ vanishing on E,*

$$\|g\|_q \leq C \left\| \frac{\omega}{d^{n-1}} |\nabla g| \right\|_q.$$

Proof of Lemma: assume first that (P_q) holds and let E and g be as in the statement of the lemma. Notice that

$$
\begin{aligned}
|E| \, |g_\Omega|^q &= \int_E |g(x) - g_\Omega|^q \, dx \\
&\leq \int_\Omega |g(x) - g_\Omega|^q \, dx \\
&\leq C \int_\Omega \left| \frac{\omega(x)}{d^{n-1}(x)} \nabla g(x) \right|^q dx,
\end{aligned}
$$

which shows that

$$|g_\Omega| \leq C' \left(\frac{1}{|\Omega|} \int_\Omega \left| \frac{\omega(x)}{d^{n-1}(x)} \nabla g(x) \right|^q dx \right)^{\frac{1}{q}}.$$

Since $\|g\|_q \leq \|g - g_\Omega\|_q + |\Omega|^{\frac{1}{q}} |g_\Omega|$, one gets the desired conclusion.

For the converse, take $g \in L^q(\Omega)$. Observe first that there exists $\lambda \in \mathbb{R}$ such that, if

$$E_\lambda := \{x \in \Omega; \ g(x) \geq \lambda\} \ \text{ and } \ F_\lambda := \{x \in \Omega; \ g(x) \leq \lambda\},$$

then

$$|E_\lambda| \geq \frac{1}{2} |\Omega| \ \text{ and } \ |F_\lambda| \geq \frac{1}{2} |\Omega|.$$

Indeed, for all $\lambda \in \mathbb{R}$, set

$$\mu(\lambda) := \{x \in \Omega; \ g(x) \leq \lambda\}.$$

The function μ is non-decreasing, right-continuous and satisfies

$$\lim_{\lambda \to -\infty} \mu(\lambda) = 0 \ \text{ and } \ \lim_{\lambda \to +\infty} \mu(\lambda) = |\Omega|.$$

Therefore, $\lambda := \inf \left\{ t \in \mathbb{R}; \ \mu(t) \geq \frac{1}{2} |\Omega| \right\}$ satisfies the required properties.
Observe that $(g - \lambda)_+ \in E^q(\Omega)$ (this can be proved by approximation arguments analogous to those used in the theory of usual Sobolev spaces). Since $(g - \lambda)_+$ vanishes on F_λ, the assumption yields

$$\|(g - \lambda)_+\|_q \leq C \left\| \frac{\omega}{d^{n-1}} |\nabla g| \right\|_q.$$

Similarly, since $(g - \lambda)_-$ vanishes on E_λ,

$$\|(g - \lambda)_-\|_q \leq C \left\| \frac{\omega}{d^{n-1}} |\nabla g| \right\|_q,$$

and we therefore obtain

$$\|g - \lambda\|_q \leq C \left\| \frac{\omega}{d^{n-1}} |\nabla g| \right\|_q,$$

from which (P_q) readily follows. □

Remark 3.5 *Lemma 3.4 extends straightforwardly to the case when E is any non-negligible measurable subset of Ω, the constant C depending on the ratio $\frac{|\Omega|}{|E|}$.*

Proof of Proposition 3.3: assume that (P_1) holds, let $q \in [1, +\infty[$ and $g \in E^q(\Omega)$ vanishing on a subset $E \subset \Omega$ satisfying $|E| \geq \frac{1}{2} |\Omega|$. Again, it can be proved as for usual Sobolev spaces that $|g|^q \in E_1(\Omega)$. Applying then (P_1) to $|g|^q$, which also vanishes on E, and using Lemma 3.4, we obtain

$$\begin{aligned}
\int_\Omega |g(x)|^q \, dx &\leq C \int_\Omega \frac{\omega(x)}{d^{n-1}(x)} |\nabla (|g|^q)| (x) dx \\
&= Cq \int_\Omega \frac{\omega(x)}{d^{n-1}(x)} |g(x)|^{q-1} |\nabla g(x)| \, dx \\
&\leq Cq \left(\int_\Omega |g(x)|^q \, dx \right)^{1-\frac{1}{q}} \left(\int_\Omega \left(\frac{\omega(x)}{d^{n-1}(x)} \right)^q |\nabla g(x)|^q \, dx \right)^{\frac{1}{q}},
\end{aligned}$$

which yields exactly (P_q) by Lemma 3.4 again. □

4 Solvability of the divergence in weighted Sobolev spaces

We now solve the divergence equation in weighted Sobolev spaces. For $p \in (1, +\infty)$, we define

$$W^{1,p} \left(\Omega, \frac{d^n}{\omega} \right) := \left\{ g \in L^p \left(\Omega, \frac{d^n}{\omega} \right); \ \partial_i g \in L^p \left(\Omega, \frac{d^n}{\omega} \right) \text{ for all } 1 \leq i \leq n \right\}$$

and $W_0^{1,p} \left(\Omega, \frac{d^n}{\omega} \right)$ stands for the closure of $\mathcal{D}(\Omega)$ in $W^{1,p} \left(\Omega, \frac{d^n}{\omega} \right)$.

Theorem 4.1 *Let $p \in (1, +\infty)$. Then, for all $f \in L_0^p(\Omega)$, there exists $u \in W_0^{1,p}(\Omega, \frac{d^n}{\omega})$ such that*

$$\operatorname{div} u = f \in \Omega$$

and

$$\|Du\|_{L^p \left(\Omega, \frac{d^n}{\omega} \right)} \leq C \|f\|_{L^p(\Omega)}, \tag{4.1}$$

where the constant $C > 0$ only depends on Ω and p.

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In the statement of Theorem 4.1, by Du, we mean the matrix $(\partial_i u_j)_{1 \leq i,j \leq n}$, and the estimate (4.1) means that

$$\sum_{1 \leq i,j \leq n} \|\partial_i u_j\|_{L^p\left(\Omega, \frac{d^n}{\omega}\right)} \leq C \|f\|_{L^p(\Omega)}.$$

The proof of Theorem 4.1 goes through a decomposition of functions in $L_0^p(\Omega)$, interesting in itself, which follows from the solvability of the divergence problem established in the previous sections:

Proposition 4.2 *Consider the Whitney decomposition of Ω already used in Section 2. Let $p \in (1, +\infty)$ and $f \in L_0^p(\Omega)$. Then, the function f can be decomposed as*

$$f = \sum_j f_j, \tag{4.2}$$

where, for all j, the function f_j satisfies the following three properties:

(i) *f_j is supported in $2Q_j$,*

(ii) *$\int_{2Q_j} f_j(x)dx = 0$,*

(iii) *$\sum_j \int_{2Q_j} |f_j(x)|^p \left(\frac{d^n(x)}{\omega(x)}\right)^p dx \leq C \int_\Omega |f(x)|^p dx$,*

for some $C > 0$ only depending on Ω, p and the choice of the paths γ.
If one furthermore assumes that there exists $C > 0$ such that the weight ω satisfies:

$$\omega(x) \leq Cd^n(x) \text{ for almost all } x \in \Omega, \tag{4.3}$$

then the conclusion of (iii) can be strengthened as

$$C^{-1} \int_\Omega |f(x)|^p dx \leq \sum_j \int_{2Q_j} |f_j(x)|^p dx \leq C \int_\Omega |f(x)|^p dx. \tag{4.4}$$

Proof: Let $f \in L^p(\Omega)$ with zero integral and, applying Theorem 3.1, pick up a vector field u such that $f = \text{div } u$ and $\left\|\frac{d^{n-1}}{\omega}u\right\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}$. Let $(\chi_j)_{j \geq 1}$ be a partition of unity associated with the Q_j's, *i.e.* a sequence of $\mathcal{D}(\mathbb{R}^n)$ functions satisfying $\sum_j \chi_j = 1$ on $\mathbb{R}^n$, such that, for all $j \geq 1$, $\chi_j = 1$ on Q_j, is supported in $2Q_j$ and satisfies $\|\nabla \chi_j\|_\infty \leq Cl_j^{-1}$. Setting

$$f_j := \text{div } (\chi_j u)$$

for all $j \geq 1$, one clearly has (4.2). It follows from the support property of χ_j that f_j is supported in $2Q_j$. That f_j has zero integral on Ω follows by an integration by parts on $2Q_j$

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and the fact that $\chi_j = 0$ on ∂Q_j. Finally, since $\omega(x) \geq cd^n(x)$ for all $x \in \Omega$,

$$
\begin{aligned}
\sum_{j \geq 1} \int_\Omega |f_j(x)|^p \left(\frac{d^n(x)}{\omega(x)}\right)^p dx \;\leq\; & C \sum_{j \geq 1} \int_{2Q_j} |\chi_j(x)|^p |f(x)|^p \left(\frac{d^n(x)}{\omega(x)}\right)^p dx \\
& + \; C \sum_{j \geq 1} \int_{2Q_j} |u(x)|^p |\nabla \chi_j(x)|^p \left(\frac{d^n(x)}{\omega(x)}\right)^p dx \\
\leq\; & C \int_\Omega |f(x)|^p \, dx + C \sum_{j \geq 1} l_j^{-p} \int_{2Q_j} |u(x)|^p \left(\frac{d^{n-1}(x)}{\omega(x)}\right)^p dx \\
\leq\; & C \int_\Omega |f(x)|^p \, dx,
\end{aligned}
$$

which is (iii).

Assume now that (4.3) holds. Then,

$$
c \leq \frac{d^n(x)}{\omega(x)} \leq C,
$$

and

$$
\int_\Omega |f(x)|^p \, dx \leq C \sum_j \int_{2Q_j} |f_j(x)|^p \, dx,
$$

which completes the proof of Proposition 4.2. $\qquad\square$

Remark 4.3 *An analogous decomposition for functions in $L^p(\Omega)$ with zero integral was established by Diening, Ruzicka and Schumacher in [10]. Actually, in their result, f can be taken in $L^p(\Omega, w)$ where the weight w belongs to the Muckenhoupt class A_q for some $q \in (1, +\infty)$, and (4.4) holds with the Lebesgue measure in both sides on the inequalities, but the authors assume that Ω satisfies an emanating chain condition (whereas no assumption is made on Ω in Proposition 4.2). The proof in [10] is direct, which means that it does not use the divergence operator, but, as in the present paper, some consequences are derived for the solvability of the divergence operator in Sobolev spaces.*

Proof of Theorem 4.1: let $f \in L^p(\Omega)$ with zero integral and decompose $f = \sum f_j$ as in Proposition 4.2. For each j, there exists $u_j \in W_0^{1,p}(2Q_j)$ such that div $u_j = f_j$ and $\|Du_j\|_{L^p(2Q_j)} \leq C \|f_j\|_{L^p(2Q_j)}$ (see for instance [7], Section 7.1, Theorem 2). Define $u := \sum_j u_j$. One clearly has div $u = f$. Moreover, by construction, there exist $0 < c < C$ with the following property: for $j \geq 1$, there exists $\omega_j \in (0, +\infty)$ such that, for all $x \in Q_j$,

$$
c\omega_j \leq \omega(x) \leq C\omega_j.
$$

As a consequence,

$$
\begin{aligned}
\int_\Omega |Du(x)|^p \left(\frac{d^n(x)}{\omega(x)}\right)^p dx \;&\leq\; C \sum_j \int_{2Q_j} |Du_j(x)|^p \left(\frac{d^n(x)}{\omega(x)}\right)^p dx \\
&\leq\; C \sum_j \frac{l_j^{np}}{\omega_j^p} \int_{2Q_j} |Du_j(x)|^p\, dx \\
&\leq\; C \sum_j \frac{l_j^{np}}{\omega_j^p} \int_{2Q_j} |f_j(x)|^p\, dx \\
&\leq\; C \int_{2Q_j} |f_j(x)|^p \left(\frac{d^n(x)}{\omega(x)}\right)^p dx \\
&\leq\; C \int_\Omega |f(x)|^p\, dx. \qquad \square
\end{aligned}
$$

As a corollary of Theorem 4.1, we obtain:

Corollary 4.4 *Let $p \in (1,n)$ and set $p^* := \frac{np}{n-p}$. Then, for all $f \in L_0^p(\Omega)$, there exists $u \in L^{p^*}\left(\Omega, \frac{d^n}{\omega}\right)$ such that*

$$
div\, u = f \in \Omega
$$

and

$$
\|u\|_{L^{p^*}\left(\Omega, \frac{d^n}{\omega}\right)} \leq C\, \|f\|_{L^p(\Omega)}, \tag{4.5}
$$

where the constant $C > 0$ only depends on Ω and p.

Proof: consider again $f \in L^p(\Omega)$ with zero integral and let f_j, u_j and u be as in the proof of Theorem 4.1. For all $j \geq 1$, by the usual Sobolev embedding,

$$
\int_{2Q_j} |u_j(x)|^{p^*} dx \leq C \left(\int_{2Q_j} |Du_j(x)|^p\, dx\right)^{\frac{p^*}{p}} \leq C \left(\int_{2Q_j} |f_j(x)|^p\, dx\right)^{\frac{p^*}{p}}.
$$

It follows that

$$
\begin{aligned}
\int_\Omega |u(x)|^{p^*} \left(\frac{d^n(x)}{\omega(x)}\right)^{p^*} dx \;&\leq\; C \sum_j \int_{2Q_j} |u_j(x)|^{p^*} \left(\frac{d^n(x)}{\omega(x)}\right)^{p^*} dx \\
&\leq\; C \sum_j \frac{l_j^{np^*}}{\omega_j^{p^*}} \int_{2Q_j} |u_j(x)|^{p^*} dx \\
&\leq\; C \sum_j \frac{l_j^{np^*}}{\omega_j^{p^*}} \left(\int_{2Q_j} |f_j(x)|^p\, dx\right)^{\frac{p^*}{p}} \\
&\leq\; C \sum_j \left(\int_{2Q_j} |f_j(x)|^p \left(\frac{d^n(x)}{\omega(x)}\right)^{p^*} dx\right)^{\frac{p^*}{p}} \\
&\leq\; C \left(\sum_j \int_{2Q_j} |f_j(x)|^p \left(\frac{d^n(x)}{\omega(x)}\right)^{p^*} dx\right)^{\frac{p^*}{p}} \\
&\leq\; C \left(\int_\Omega |f(x)|^p\, dx\right)^{\frac{p^*}{p}},
\end{aligned}
$$

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which ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5   Examples

In this section we give various examples of domains Ω to which the preceding results apply and for which we obtain an estimate of the weight w.

In their paper [1], the authors show how to invert the divergence for John domains, with the weight $\mathrm{w}(x) = d(x)$ (note that the corresponding weighted Poincaré inequalities on John domains were established in [11]). Our method gives a generalization to the so-called s-John domains, that we will present first; then we will turn to a particular case which exhibits special features, that of strongly hölderian domains.

5.1   The case of s-John domains

Let $s \geq 1$ be a fixed parameter. Recall that Ω is an s-John domain when, x_0 being a chosen reference point, there exist a constant $C > 0$ and, for each $y \in \Omega$, a rectifiable path $\gamma(y)$ included in Ω which links y to x_0 and satisfies the estimate

$$\forall \tau \in [0, l(\gamma(y))] \ d(\gamma(\tau, y)) \geq C\tau^s, \qquad (5.1)$$

where τ is the arc-length and $l(\gamma(y))$ the total length of the path $\gamma(y)$. When $s = 1$, Ω is a John domain. Using the same algorithm as in the proof of Theorem 2.1, we modify the initial family γ so as to ensure properties $(\gamma.a)$ to $(\gamma.d)$. We let the reader check that property (5.1) remains preserved. Here are explicit examples.

Example 1 Let Ω be the complement in $B(0, 10)$ of the logarithmic spiral

$$\Gamma = \{x = r(\cos\theta, \sin\theta); r = 0 \text{ or } r \in]0, 1] \text{ with } r = e^{-\theta}\}.$$

Then Ω is a John domain.

Example 2 Let Ω be the complement in $B(0, 10)$ of the spiral

$$\Gamma = \{x = r(\cos\theta, \sin\theta); r = 0 \text{ or } r \in]0, 1] \text{ with } \theta = r^{-a}\}$$

for some $a > 0$. Then Ω is an s-John domain if and only if $a < 1$, with $s = \frac{1+a}{1-a}$. Note, however, that Ω fulfills (2.4) if and only if $a < 3$.

Returning to the general s-John domains, we now assume that Ω fulfills hypothesis (2.4). Applying Lemma 2.3 and the preceding sections, we can invert the divergence on various L^p spaces on Ω with the weight

$$\mathrm{w}(x) = \omega(x)d(x)^{-n+1},$$

where ω is defined by (2.7). We then have:

Lemma 5.1 $\omega(x) \leq Cd(x)^{\frac{n}{s}}$.

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Hence we recover the result of [15], Theorem 7, that on s-John domains the following Poincaré inequality holds:

$$\int_\Omega |g(x) - g_\Omega|\, dx \le C(\Omega) \int_\Omega d(x)^{\frac{n}{s} - n + 1} \, |\nabla g(x)| \, dx.$$

It is known, also, that $\frac{n}{s} - n + 1$ is the best possible exponent in the class of s-John domains (see [5]). Let us, however, remark that the estimate in this Lemma may be of poor quality, since it may happen for large values of s that

$$\int_0^{+\infty} d(x)^{\frac{n}{s} - n + 1} dx = +\infty,$$

while we know that $\mathrm{w} \in L^1(\Omega)$. This reveals that, at least in some cases, the family of weights $d(x)^\alpha$, $\alpha \in \mathbb{R}$, is not rich enough to accurately describe what happens.

The proof of Lemma 5.1 is elementary. Let $x, y \in \Omega$ such that $\mathrm{dist}_\Omega(\gamma(y), x) \le \frac{1}{2} d(x)$. There exists $\tau \in [0, l(\gamma(y))]$ verifying

$$|\gamma(\tau, y) - x| \le \frac{1}{2} d(x),$$

and thus

$$\tau^s \le C d(\gamma(\tau, y)) \le C d(x).$$

Since $|y - x| \le \tau + \frac{1}{2} d(x)$, we have $y \in B(x, C d(x)^{\frac{1}{s}})$, which implies $\omega(x) \le C d(x)^{\frac{n}{s}}$ as desired.

5.2   The case of strongly hölderian domains

There is, however, a particular case where the above estimate can be improved as soon as $s > 1$. This is when Ω, in addition to the preceding hypothesis, fulfills the requirements of the following definition.

Definition 5.2 *Say that Ω is a strongly hölderian domain when there exist $\alpha \in]0, 1]$ and an integer $N \ge 1$, N functions $\Phi_j \in C^\alpha(\mathbb{R}^{n-1})$, $N + 1$ domains O_j in $\mathbb{R}^n$ and N isometries η_j of $\mathbb{R}^n$ such that*

· $\Omega \subset \bigcup_{j=0}^N O_j$,

· $\overline{O_0} \subset \Omega$,

· *when $x \in O_j$, $j \ge 1$, then $x \in \Omega$ if and only if $\widetilde{x_n} < \Phi_j(\widetilde{x'})$, where $\eta_j(x) = (\widetilde{x'}, \widetilde{x_n}), \widetilde{x'} \in \mathbb{R}^{n-1}, \widetilde{x_n} \in \mathbb{R}$,*

· *there exists $h > 0$ such that, if $x \in O_j \cap \Omega, j \ge 1$, and $\widetilde{x_n} < \Phi_j(\widetilde{x'}) - h$, then $x \in O_0$,*

· *conversely, for any $x \in O_j \cap \Omega, j \ge 1$, there exists $z \in O_0$ such that $\widetilde{z'} = \widetilde{x'}$ and $\widetilde{z_n} \le \widetilde{x_n}$, where $\eta_j(z) = (\widetilde{z'}, \widetilde{z_n})$.*

We construct a family γ of paths satisfying properties $(\gamma.a)$ to $(\gamma.d)$ and adapted to the nature of the boundary of Ω.

Choose $x_0 \in O_0$ and let $y \in \Omega$. When $y \in O_0$, we cover O_0 by a finite number of Whitney cubes associated to Ω and apply the algorithm described in the proof of Theorem 2.1 to obtain a family of paths $\gamma(y)$ which lie inside O_0.

When $y \in O_j, j \geq 1$, we assume for simplicity that $\eta_j = Id$ (which is possible by rotating Ω). Write $y = (y', y_n)$ and select $z = (y', z_n) \in O_0$ with $z_n \leq y_n$. We link y to z by the vertical segment $\{(y', (1-t)y_n + tz_n); t \in [0,1]\}$, and then z to x_0 as described above.

The reader may check that the resulting family of paths satisfies properties $(\gamma.a)$ to $(\gamma.d)$, and Ω is an s-John domain for $s = \frac{1}{\alpha}$.

Let w be the associated weight, which allows to solve the problem $(\text{div})_\infty$ in Ω thanks to Lemma 2.3. We have the following improvement over the result of Lemma 5.1:

Lemma 5.3 *There exists a constant $C > 0$ such that*

$$\forall x \in \Omega, \ w(x) \leq Cd(x)^\alpha.$$

Since d^α is always integrable over Ω, this Lemma implies that, for this class of domains, the problem $(\text{div})_\infty$ is solvable with the weight d^α.

Let us turn to the proof and take $x, y \in \Omega$ with $\text{dist}_\Omega(\gamma(y), x) \leq \frac{1}{2}d(x)$. We may assume that $d(x) < \delta$ for some $\delta > 0$ to be chosen, since otherwise there is nothing to prove. If δ is small enough, there exists $j \geq 1$ such that $x, y \in O_j$. We again simplify the argument by assuming $\eta_j = Id$. Write $y = (y', y_n)$ and $z = (y', z_n)$ as above. The distance from $\gamma(y)$ to x is not attained at a point lying in $\overline{O_0}$ provided δ is sufficiently small, so that there exists $t \in [0, 1[$ satisfying

$$|(y', (1-t)y_n + tz_n) - x| \leq \frac{1}{2}d(x).$$

Hence we have

$$|y' - x'| \leq \frac{1}{2}d(x) \tag{5.2}$$

and (recall that $z_n < y_n$)

$$x_n - \frac{1}{2}d(x) \leq y_n < \Phi_j(y'). \tag{5.3}$$

We want to show that

$$|y_n - x_n| \leq Cd(x)^\alpha. \tag{5.4}$$

From (5.2) and the α-regularity of Φ_j, we deduce

$$|\Phi_j(y') - \Phi_j(x')| \leq Cd(x)^\alpha$$

and

$$y_n < \Phi_j(x') + Cd(x)^\alpha. \tag{5.5}$$

Let $(u', \Phi_j(u'))$ be a point of $\partial\Omega$ at which $d(x)$ is attained (such a point exists once δ is small enough and j is appropriately chosen). We have $|x' - u'| \leq d(x)$ and $|\Phi_j(u') - x_n| \leq d(x)$, whence

$$\begin{aligned} |\Phi_j(x') - x_n| &\leq |\Phi_j(x') - \Phi_j(u')| + d(x) \\ &\leq Cd(x)^\alpha. \end{aligned}$$

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From (5.3) and (5.5) we then have

$$x_n - \frac{1}{2}d(x) \le y_n \le x_n + Cd(x)^\alpha,$$

which proves (5.4). With (5.2), this implies

$$\omega(x) \le Cd(x)^{n-1+\alpha},$$

and the proof is readily ended.

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